on amicable numbers in Dickson's *History of the Theory of Numbers* and seeing how large a number of papers had been written on the subject and how relatively few numbers had been found. After some time spent in searching he is convinced that any one with a little skill in manipulation of numbers, considerable patience, and access to the Lehmer's *List of Prime Numbers,* can add to the known list of amicable number pairs, amicable k-tuples or multiply amicable number sets.

AN ELEMENTARY TREATMENT OF FOURIER'S SERIES.

By GEORGE D. BIRKHOFF,[1] Harvard University.

The aim of this note is to treat the "remainder" after $n + 1$ terms of the Fourier's series for a given function $f(x)$:

$$\tfrac{1}{2}a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots,$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \cos kx dx, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x) \sin kx dx.$$

It will be assumed that the function $f(x)$ is periodic of period 2π and continuous together with its first three derivatives.

Let $T_n(x)$ stand for the sum of the first $n + 1$ terms of the above series; its derivative $T_n'(x)$ is seen at once not to exceed in numerical value

$$(|a_1| + |b_1|) + 2(|a_2| + |b_2|) + \cdots + n(|a_n| + |b_n|).$$

Furthermore, by integration by parts and use of the periodicity of $f(x)$, we find

$$a_k = -\frac{1}{\pi k}\int_{-\pi}^{+\pi} f'(x) \sin kx dx = -\frac{1}{\pi k^2}\int_{-\pi}^{+\pi} f''(x) \cos kx dx$$

$$= \frac{1}{\pi k^3}\int_{-\pi}^{+\pi} f'''(x) \sin kx dx.$$

Consequently $|a_k|$, and similarly $|b_k|$, is less than $2F_3/k^3$ where F_3 is the maximum of $|f'''(x)|$. We conclude that $|T_n'(x)|$ is not greater than

$$4F_3\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) < 4F_3\left(1 + \left(1 - \frac{1}{2}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right),$$

[1] Professor Birkhoff's earlier paper in this MONTHLY, "Note on certain quadratic number systems for which factorization is unique," appeared almost exactly fifteen years ago. His first published paper, in collaboration with H. S. Vandiver, "On the integral divisors of $a^n - b^n$," was published in *Annals of Mathematics*, 1904. His second paper appeared also in *Annals . . .*, 1905. His third paper was a thirty page memoir in *Transactions of the American Mathematical Society*, 1906; and his fourth paper is the one referred to above. A complete collection of his mathematical papers 1904–1919, 37 in number, is preserved in a bound volume at the mathematical seminary of Brown University.—EDITOR.

so that

(1) $$|T_n'(x)| < 8F_3.$$

The expression $T_n(x)$ has also the properties

(2) $$\begin{cases} a_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} T_n(x)\cos kx dx, \\ b_k = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{+\pi} T_n(x)\sin kx dx, \end{cases}$$

for $k \leqq n$, as follows by direct integration. Using the equations of definition of a_k, b_k in combination with (2) we deduce at once

(3) $$\int_{-\pi}^{+\pi} R_n(x)\cos kx dx = \int_{-\pi}^{+\pi} R_n(x)\sin kx dx = 0, \qquad (k \leqq n),$$

where $R_n(x)$ denotes the "remainder" $f(x) - T_n(x)$.

Now by means of the familiar product formulas of trigonometry we may express successively $\cos^2 x$, $\cos x \sin x$, $\sin^2 x$, $\cos^3 x$, $\cdots$, $\sin^n x$ as linear combinations of 1, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$, $\sin nx$. Therefore we find from (3)

(3′) $$\int_{-\pi}^{+\pi} R_n(x)\cos^\alpha x \sin^\beta x dx = 0, \qquad (\alpha + \beta \leqq n),$$

and, noting the equation

$$\cos^2\frac{x - x_0}{2} = \tfrac{1}{2}(1 + \cos x \cos x_0 + \sin x \sin x_0),$$

we infer from (3′)

(4) $$\int_{-\pi}^{+\pi} R_n(x)\cos^{2n}\frac{x - x_0}{2} dx = 0$$

for any x_0. For convenience we make the change of variables $x = x_0 + t$ and obtain

(4′) $$\int_{-\pi}^{+\pi} R_n(x_0 + t)\cos^{2n}\frac{t}{2} dt = 0.$$

The limits can be made $\pm\pi$ since the integrand is periodic of period 2π in t.

By the aid of (4′) we propose to obtain an upper limit for the "remainder" $R_n(x_0)$. We have

$$\begin{aligned} |R_n(x_0 + t) - R_n(x_0)| &\leqq |f(x_0 + t) - f(x_0)| + |T_n(x_0 + t) - T_n(x_0)| \\ &\leqq (F_1 + 8F_3)|t|, \end{aligned}$$

where F_1 stands for the maximum of $|f'(x)|$ and $8F_3$ is at least as great as

$|T_n'(x)|$ by (1). Now the ratio

$$\frac{y}{\sin y} \qquad \left(|y| \leqq \frac{\pi}{2}\right)$$

is positive and may be seen geometrically or otherwise to have its maximum value $\pi/2$ for $y = \pm\,\pi/2$. Therefore it is clear that

$$|t| \leqq \pi \left|\sin\frac{t}{2}\right|, \qquad (|t| \leqq \pi),$$

whence we find

$$(5) \qquad |R_n(x_0 + t) - R_n(x_0)| \leqq N|\sin \tfrac{1}{2}t|, \qquad (N = \pi(F_1 + 8F_3)).$$

Thus we may substitute in (4′)

$$(5') \qquad R_n(x_0 + t) = R_n(x_0) + \theta N \sin \tfrac{1}{2}t$$

where $|\theta| \leqq 1$. When we do so there results

$$(6) \qquad R_n(x_0) = -\frac{\displaystyle\int_{-\pi}^{+\pi} \theta N \cos^{2n}\frac{t}{2}\sin\frac{t}{2}\,dt}{\displaystyle\int_{-\pi}^{+\pi}\cos^{2n}\frac{t}{2}\,dt}.$$

The numerator on the right in (6) is not greater numerically than

$$N\int_{-\pi}^{+\pi}\cos^{2n}\frac{t}{2}\left|\sin\frac{t}{2}\right|dt = 2N\int_0^{\pi}\cos^{2n}\frac{t}{2}\sin\frac{t}{2}\,dt = \frac{4N}{2n+1}.$$

Also the familiar formulas

$$\int_0^{\pi/2}\cos^n x\,dx = \begin{cases} \dfrac{1\cdot 3\cdots n-1}{2\cdot 4\cdots n}\cdot\dfrac{\pi}{2} & (n \text{ even}),\\[2ex] \dfrac{2\cdot 4\cdots n-1}{3\cdot 5\cdots n} & (n \text{ odd}),\end{cases}$$

show that

$$\int_0^{\pi/2}\cos^n x\,dx\cdot\int_0^{\pi/2}\cos^{n+1}x\,dx = \frac{\pi}{2(n+1)}.$$

Of the two integrals on the left the first is of course the greater, and it follows that

$$\int_0^{\pi/2}\cos^n x\,dx > \sqrt{\frac{\pi}{2(n+1)}}, \qquad (n = 0, 1, \cdots),$$

and thus that

$$\int_{-\pi}^{+\pi}\cos^{2n}\frac{t}{2}\,dt = 4\int_0^{\pi/2}\cos^{2n}x\,dx > 2\sqrt{\frac{2\pi}{2n+1}}.$$

If we replace the numerator and denominator in (6) by the greater and lesser values respectively which have now been found, the fraction is increased numer-

ically; whence finally

$$|R_n(x)| < N\sqrt{\frac{2}{\pi(2n+1)}} \tag{7}$$

for every value x_0 of x and for every $n \geqq 0$. Thus the remainder approaches 0 as n increases indefinitely, and the Fourier's series converges to $f(x)$.

The method outlined above may easily be modified so as to deal with a more general type of function $f(x)$, for instance any periodic function with two derivatives. Essentially the only novel feature of the method is the direct use of an equation like (4) to obtain an upper limit for the remainder.[1]

A CUBIC SPACE CURVE CONNECTED WITH THE TETRAHEDRON.

By FRANCIS D. MURNAGHAN, Johns Hopkins University.

In connection with the geometry of the triangle there is a well known circumconic, known as Kiepert's hyperbola,[2] which passes through the orthocenter and centroid of the triangle. In the following note an analogue of this hyperbola for the tetrahedron is given, and it is hoped that interest may thereby be aroused in the still comparatively unexplored field of the "geometry of the tetrahedron." [3]

It will be convenient to recall an interesting property of Kiepert's hyperbola—a property which may be used to define the curve and at the same time to give a simple parametric representation of the points on it. Let A_1, A_2, A_3 be the triangle and on each of the segments A_2A_3, A_3A_1, A_1A_2 describe a circle containing an angle θ; θ being in each case the angle subtended by the segment at points of that arc of the corresponding circle which is on the same side of the segment as the opposite vertex, so that θ can vary from 0° to 180°. For any given value of θ the three circles have a radical center and Kiepert's hyperbola is the locus of these radical centers. In order to find the coördinates of the radical center for a given value of θ, it is probably simplest to introduce, momentarily, rectangular axes with origin at O_1, the mid-point of A_2A_3, and with the X axis along A_2A_3. If $2a_1$ is the length of A_2A_3, the coördinates of C_1, the center of the circle through A_2A_3, are $(0, a_1 \cot\theta)$ and the radius is $a_1 \operatorname{cosec}\theta$, so that the equation is

---

[1] See Lebesgue, *Leçons sur les séries trigonométriques*, Paris, 1906, pp. 37–38, and de la Vallée Poussin, *Bulletins de l'Académie royale de Belgique*, classe des sciences, 1908, pp. 193–254, in particular p. 230 *et seq.*

[2] Kiepert, *Nouvelles Annales de Mathématiques*, 1869, p. 42. Other discussions of this hyperbola are: by Brocard, in *Journal de Mathématiques Spéciales*, 1884, pp. 197–209 and 1885, pp. 12–15, 30–33, 58–64, 76–80, 104–112, 123–131; by de Longchamps, *idem*, 1886, pp. 77–79 231–235; by M'Cay, in *Mathesis*, 1887, pp. 208–220; by Laisant, in *Compte rendu . . . Association Française pour l'Avancement des Sciences*, seconde partie, 1887, pp. 113–114; by J. Casey, in his *A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections*, second edition, 1893, pp. 431, 442–445, 449, 452, 453; and by I. J. Schwatt, in his *A geometrical Treatment of Curves which are isogonal conjugate to a straight line with respect to a triangle.* Part 1, Boston, New York, and Chicago, [c. 1895], pp. 3–28.—EDITOR.

[3] Those interested may be referred to two papers by J. Neuberg in *Archiv der Mathematik und Physik*, 3. Reihe, vols. 16 and 18, 1910–1911.